Stability of synchronized states in networks of phase oscillators

Keivan Aghababaei Samani* and Shahzad Ghanbarian†
Department of Physics, Isfahan University of Technology, Isfahan 84156-83111, Iran
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The stability of synchronized states (frequency locked states) in networks of phase oscillators is investigated for several network topologies. It is shown that for some topologies there is more than one stable synchronized state according to the sign of coupling strength between oscillators. It is also shown that in some cases the synchronized state corresponds to zero order parameter.

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I. INTRODUCTION

Synchronization of a population of interacting oscillators has been the subject of many investigations in recent years [1]. The motivation for such research comes from the wide range of examples and applications of synchronization in real world. Synchronization phenomenon is a property of several biological, physical, chemical, and social systems [2–4].

The mathematical model for synchronization of phase oscillators was proposed by Winfree [5] and Kuramoto [6,7]. In this paper we follow the Kuramoto model in which the phase oscillators are coupled nonlinearly to each other (through a sine function).

One of the interesting questions in the Kuramoto model is the stability of solutions, both synchronized and incoherent states. In this paper, by a *synchronized state* we mean a state in which all of the oscillators oscillate with the same frequency (this is also called a *frequency locked state*). The question of stability is addressed by Kuramoto himself and many authors for several physical cases such as stability in presence of noise [1,9,10], stability of coupled identical oscillators [11,12], stability of all to all coupled oscillators with arbitrary frequency distribution [8], and stability of globally coupled oscillators with nonsinusoidal coupling [3]. Here we investigate the stability of synchronized state in three kinds of networks, namely all-to-all network, bipartite network, and semibipartite network [13]. In each of these networks we consider two cases for frequency distribution: the case of identical oscillators and the case of two types of oscillators which are called *unimodal* and *bimodal* frequency distributions, respectively. A bipartite network of oscillators is important when we deal with oscillators of two different kinds and couplings are present only between oscillators of different kinds. Another importance of bipartite networks is the case of lattices. Some lattices, such as two dimensional square lattice and honeycomb, are bipartite graphs. If only nearest neighbor couplings are considered, then they are also a bipartite network of coupled oscillators. These are especially important in studying lattices of Josephson junctions and laser arrays [1]. A semibipartite network is a model for a network with a highly connected center. Such networks are of importance in studying neural networks, biological neural systems, and models of memory [14]. The results obtained in this paper are exact and independent of the system size.

The organization of the paper is as follows. In Sec. II we introduce the Kuramoto model for an arbitrary network and give the definition of frequency locked state and the method of checking the stability. In Secs. III and IV we explore the cases of unimodal and bimodal frequency distributions in the above mentioned networks, respectively. Section V is devoted to summary and concluding remarks.

II. THE KURAMOTO MODEL ON AN ARBITRARY NETWORK

The Kuramoto model on a network of N phase oscillators is given by the following equations:

$$\dot{\theta}_i = \omega_i + \sum_{j=1}^{N} K_{ij} \sin(\theta_j - \theta_i), \quad i = 1, \ldots, N, \tag{1}$$

where K_{ij} is the coupling strength between ith and jth oscillators which is assumed to be symmetric, i.e., $K_{ij}=K_{ji}$. A solution of Eq. (1) will be denoted by $\theta_i^*, i=1,2,\ldots,N$. It should be mentioned that the solution is not necessarily unique. Furthermore, a solution may correspond to a synchronized state or an incoherent state. In this paper we will focus on solutions which correspond to synchronized states. As mentioned in the Introduction, a *synchronized state* is one in which all of the oscillators are of the same frequency. We are going to investigate the stability of this solution. To this end we perturb the solution slightly, i.e., $\theta_i=\theta_i^*+\epsilon_i$, and then find the equation of motion for ϵ_is to the first order and check the Lyaponov exponents, i.e., eigenvalues of the matrix of coefficients of ϵ_is. The solution is stable if there is no positive exponent. It is straightforward to see that Eq. (1) leads to the following equation for ϵ_is:

$$\dot{\epsilon}_i = \sum_{j=1}^{N} K_{ij} \cos(\theta_j^* - \theta_i^*)(\epsilon_j - \epsilon_i), \tag{2}$$

which may be written in a more elegant way:

$$\dot{\epsilon} = \Lambda \epsilon, \tag{3}$$

where $\epsilon=(\epsilon_1,\epsilon_2,\ldots,\epsilon_N)^t$ and $\Lambda=M-D$ with $M_{ij}=K_{ij}\cos(\theta_j^*-\theta_i^*)$ and $D=\mathrm{diag}(d_1,d_2,\ldots,d_N)$ where $d_i=\Sigma_{j=1}^{N}M_{ij}$. If none of the eigenvalues of Λ are positive then

*samani@cc.iut.ac.ir
†sh_ghanbarian@ph.iut.ac.ir

the solution is stable. It may seem strange that in the stability criterion there is no ω_i. In fact, the effect of ω_is can be traced in the solution θ_i^* which of course depends on ω_is. In subsequent sections we will use this criterion to investigate the stability of a synchronized state for three networks, namely all-to-all network, bipartite network, and semibipartite network. As for the frequency distribution we take two cases: identical oscillators and two kinds of oscillators.

Throughout this paper we will assume that $K_{ij}=\frac{K}{N}\mathcal{A}_{ij}$ where $\mathcal{A}$ is the adjacency matrix of the network. This means that coupling strength is identical for all couples of linked oscillators. To fix the notation we define the Laplacian matrix of a network by $\mathcal{L}=\mathcal{D}-\mathcal{A}$ where $\mathcal{D}$ is a diagonal matrix with $\mathcal{D}_{ii}=\Sigma_{j=1}^{N}\mathcal{A}_{ij}$. The Laplacian matrix of any network is *semipositive definite*, i.e., its eigenvalues are non-negative. This is shown in the Appendix.

III. UNIMODAL FREQUENCY DISTRIBUTION

An oscillator is identified by its intrinsic frequency. Therefore the term *identical oscillators* means that all of the oscillators have the same intrinsic frequency. In such a case one can go to a reference frame in which the intrinsic frequencies of oscillators are zero and the frequency distribution function of oscillators is $f(\omega)=\delta(\omega)$. Therefore Eq. (1) reads

$$\dot{\theta}_i=\sum_{j=1}^{N}K_{ij}\sin(\theta_j-\theta_i),\quad i=1,\ldots,N. \tag{4}$$

A synchronized state in this reference frame will be characterized by $\omega=0$. Therefore the solutions of the following equation correspond to synchronized states:

$$\sum_{j=1}^{N}K_{ij}\sin(\theta_j-\theta_i)=0,\quad i=1,\ldots,N. \tag{5}$$

The simplest solution is one with all oscillators in the same phase $\theta_i=\theta^*$, $i=1,2,\ldots,N$. We call this solution the *trivial solution*. It is easily seen that for this solution $\Lambda=-\frac{K}{N}\mathcal{L}$. As the eigenvalues of the Laplacian are non-negative, the trivial solution is stable (unstable) for $K>0$ ($K<0$). In the following subsections we try to find nontrivial solutions.

A. All-to-all network

If the network is all to all then Eq. (5) reads

$$\frac{K}{N}\sum_{j=1}^{N}\sin(\theta_j-\theta_i)=0,\quad i=1,\ldots,N. \tag{6}$$

This equation may have many solutions. To find the solutions we use the *order parameter* which is defined as

$$re^{i\psi}=\frac{1}{N}\sum_{j=1}^{N}e^{i\theta_j}. \tag{7}$$

Using this definition in Eq. (6) one arrives at

$$r\sin(\psi-\theta_i)=0,\quad i=1,\ldots,N. \tag{8}$$

Therefore there are two types of solutions: $r=0$ and $\sin(\psi-\theta_i)=0$.

Case 1: $r=0$. In this case $D=-\frac{K}{N}I$ according to Eq. (7) and therefore $\Lambda_{ij}=\frac{K}{N}\cos(\theta_j^*-\theta_i^*)$. By defining $c^t=(\cos\theta_1^*,\cos\theta_2^*,\ldots,\cos\theta_N^*)$ and $s^t=(\sin\theta_1^*,\sin\theta_2^*,\ldots,\sin\theta_N^*)$, Λ may be written as $\Lambda=\frac{K}{N}(cc^t+ss^t)$ which is clearly semipositive (seminegative) definite for $K>0$ ($K<0$). Therefore for $K<0$ these solutions are stable although the order parameter is exactly zero.

Case 2: $\sin(\psi-\theta_i)=0$. There are many nontrivial solutions in this case. In fact, each oscillator can take the phase value ψ or $\psi+\pi$. Without loss of generality a nontrivial solution may be written in the following general form:

$$\theta_i=\begin{cases}0 & 1\le i\le N_0\\ \pi & N_0+1\le i\le N\end{cases}, \tag{9}$$

and the matrix of coefficients will be

$$\Lambda=\frac{K}{N}\begin{pmatrix}[J+(N_\pi-N_0)I]_{N_0} & -L_{N_0\times N_\pi}\\ -L^t_{N_\pi\times N_0} & [J-(N_\pi-N_0)I]_{N_\pi}\end{pmatrix}, \tag{10}$$

where N_0 and N_π are the number of oscillators with zero and π phase values, respectively. The matrix J is a square matrix with all elements equal to 1. Here (say) M_N stands for the *square matrix M* which is of dimension N. The matrix L is a $N_0\times N_\pi$ matrix with all elements equal to 1 and L^t is it transposed. The matrix I is the identity matrix. Using Eq. (A2) of the Appendix the eigenvalues of Λ are

$$\mathrm{Spec}(\Lambda)=\begin{pmatrix}\frac{K}{N}(N-2N_0) & -\frac{K}{N}(N-2N_0) & K & 0\\ N_0-1 & N-N_0-1 & 1 & 1\end{pmatrix}, \tag{11}$$

where the first row are eigenvalues and the second row their multiplicity. It should be mentioned that Eq. (11) is not correct for $N_0=0$ and $N_0=N$. In fact, these two cases give the trivial solution which is already discussed. A special case is $N_\pi=N_0$. In this case all of the eigenvalues of Λ are zero except one of them which is K and therefore this solution is stable for $K<0$. In fact, in this case $r=0$ and this solution also belongs to case 1. If $N_\pi\neq N_0$ the spectrum of Λ shows that the solution is always unstable because of some positive eigenvalues, i.e., $\frac{K}{N}(N-2N_0)$ or $-\frac{K}{N}(N-2N_0)$.

B. Bipartite network

Some of the networks in real world have the structure of bipartite networks such as collaboration networks and some neural networks [15,16]. This fact has motivated several investigations on bipartite structures in complex networks in recent years. The reader may refer for example to Refs. [17,18]. In a bipartite network there are two kinds of nodes and only nodes of different kinds may connect to each other. If N_1 and N_2 are the number of two kinds of nodes, respectively, then the adjacency matrix of a bipartite network can be written in off block diagonal form,

$$A = \begin{pmatrix} 0 & L \\ L^t & 0 \end{pmatrix}, \quad (12)$$

where L is a $N_1 \times N_2$ matrix with zero and one elements.

There is an important nontrivial solution in which the oscillators in one part are in phase $\theta = 0$ and oscillators in the second part are in phase $\theta = \pi$. In this case $\Lambda = \frac{K}{N}\mathcal{L}$ and the solution is stable (unstable) for $K < 0$ $(K > 0)$.

In a bipartite network the trivial solution and the above mentioned nontrivial solution may be named as ferromagnetic (FM) and antiferromagnetic (AFM) solutions, respectively. In fact, in the trivial solution all oscillators are in phaselike spins in a ferromagnetic material, but in nontrivial solution the adjacent oscillators are in opposite phases like spins in an antiferromagnetic material. It is worth noting that depending the sign of the coupling constant (K) only one of these solutions is stable. Positive (negative) K is like an attractive (repelling) force between oscillators in agreement with FM and AFM names for trivial and nontrivial solutions.

C. Semibipartite network

A semibipartite network [13] is constructed from two parts. The first part which is called *center* of the network is completely connected, i.e., there is a link between every two nodes. This part of the network is a complete graph. The nodes of the second part are called *peripheral nodes*. There are some links between the nodes of the first part (*central nodes*) and the nodes of the second part (*peripheral nodes*), but there is no link between the nodes of the second part (peripheral nodes are not connected together directly). Such a network can be the model of a network with a highly connected center such as a computer network in which there are some *server* computers which are all connected together and some *client* computers which are connected to some of the server computers but there is no direct link between two client computers. Another example is a neural network with a center (say brain) in which almost all neurons are connected to each other and some peripheral neurons which are not connected to each other but all of them have some connections with the neurons of brain.

In this paper we only consider complete semibipartite networks where each peripheral node is connected to all central nodes. The adjacency matrix of such a network is

$$A = \begin{pmatrix} (J-I)_{N_c \times N_c} & L_{N_c \times N_p} \\ L^t_{N_p \times N_c} & 0 \end{pmatrix}. \quad (13)$$

Here J is a matrix in which all of its elements are 1 and I is the identity matrix. N_c and N_p are the number of central and peripheral nodes, respectively, and $N_c + N_p = N$.

An important nontrivial synchronized solution is

$$\theta_i = \begin{cases} 0 & 1 \le i \le N_c \\ \pi & N_c < i \le N \end{cases}. \quad (14)$$

The matrix Λ for this solution is

$$\Lambda = \frac{K}{N}\begin{pmatrix} J - (N_c - N_p)I & -L \\ -L^t & N_c I \end{pmatrix}, \quad (15)$$

where L is a $N_c \times N_p$ matrix with all elements equal to 1. The spectrum of Λ is

$$\mathrm{Spec}(\Lambda) = \begin{pmatrix} \frac{K}{N}(N - 2N_c) & \frac{K}{N}N_c & K & 0 \\ N_c - 1 & N_p - 1 & 1 & 1 \end{pmatrix}. \quad (16)$$

Therefore for $K > 0$ this solution is always unstable. For $K < 0$ the solution is stable if $N_p \ge N_c$. An important special case is $N_c = 1$ (a star). In this case some in phase oscillators are connected to a center which is in opposite phase and this state is stable because of a repelling coupling constant.

IV. BIMODAL FREQUENCY DISTRIBUTION

In this section we consider networks of two types of oscillators. Since in our formulation each oscillator is identified by its frequency, without loss of generality, one can assume that the oscillators are of frequencies ω and $-\omega$. One can do this by going to an appropriate rotating reference frame. This is called *bimodal* frequency distribution [9]. In subsequent subsections we consider three kinds of networks of Sec. II with bimodal frequency distribution.

From Eq. (1) it can be easily seen that $\Sigma \dot{\theta}_i = \Sigma \omega_i$. Therefore in a synchronized state with $\dot{\theta}_i = \Omega$ we have $\Omega = \frac{1}{N}\Sigma \omega_i =: \bar{\omega}$

A. All-to-all network

In this section we consider a complete network with two types of oscillators with frequencies ω and $-\omega$. We are seeking solutions in which all of the oscillators rotate with the same frequency say Ω. Suppose that the number of oscillators with frequency ω is N_+ and the number of oscillators with frequency $-\omega$ is N_-. We define $\alpha = \frac{N_+}{N}$ for convenience. Then if one also uses Eq. (7), the equations of motion read

$$\Omega = \begin{cases} \omega + Kr\sin(\psi - \theta_i) & 1 \le i \le N_+ \\ -\omega + Kr\sin(\psi - \theta_i) & N_+ < i \le N \end{cases}, \quad (17)$$

Equation (17) and the fact that $\Omega = (2\alpha - 1)\omega$ yield

$$\theta_i = \begin{cases} \theta_+ & 1 \le i \le N_+ \\ \theta_- & N_+ < i \le N \end{cases}, \quad (18)$$

where $\sin(\theta_+ - \theta_-) = \frac{2\omega}{K} =: \sin\gamma$. This gives an important condition for the existence of synchronized solutions: $|K| \ge K_c := 2\omega$. In Fig. 1 the time evolution of the order parameter is shown for three values of K. The synchronized state is achieved for $|K| \ge K_c$. Putting this solution into the definition of the order parameter one gets

$$r^2 = \alpha^2 + (1 - \alpha)^2 + 2\alpha(1 - \alpha)\cos\gamma. \quad (19)$$

In Fig. 2 the time evolution of the order parameter is shown for two values of $\sin\gamma$. The complete agreement with Eq. (19) is clear. Next we examine the stability of this solution. To this end we construct the matrix Λ,

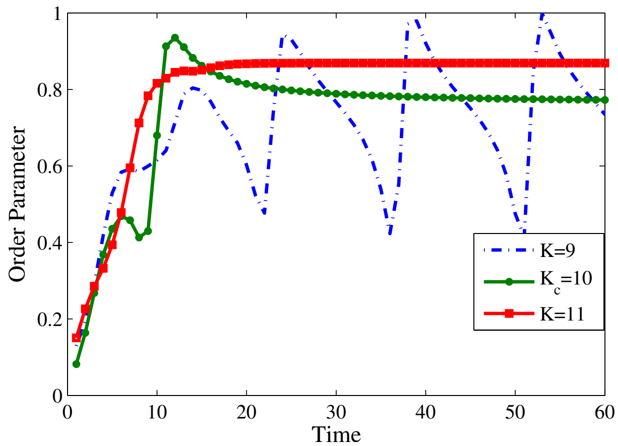

FIG. 1. (Color online) Time evolution of order parameter in an all-to-all network with 100 nodes and bimodal frequency distribution with $\omega=5$ and $\alpha=0.3$ for three values of coupling constant: $K<K_c$, $K=K_c$, and $K>K_c$.

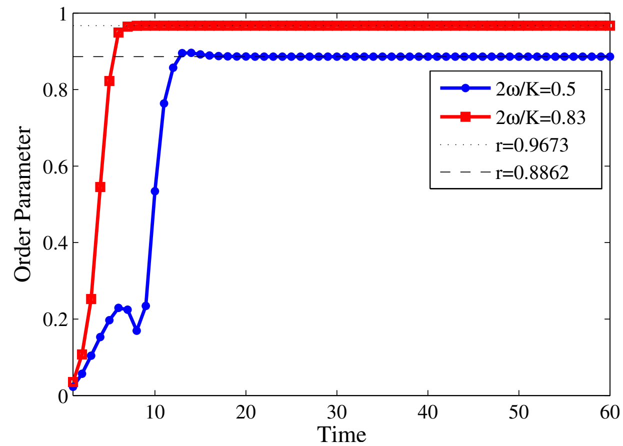

FIG. 3. (Color online) Time evolution of order parameter in a bipartite network with 40 and 60 nodes in its two parts, respectively, and bimodal frequency distribution with $\omega=5$ and $\alpha=0.4$ for two values of $\sin\,\gamma$.

$$\Lambda = \frac{K}{N}\begin{pmatrix} J-\lambda_+ I & \cos\,\gamma L \\ \cos\,\gamma L^t & J-\lambda_- I \end{pmatrix}, \tag{20}$$

where $\lambda_{\pm}=(N_{\pm}+N_{\mp}\cos\,\gamma)$. The spectrum of Λ is

$$\text{Spec}(\Lambda) = \begin{pmatrix} -\frac{K}{N}\lambda_+ & -\frac{K}{N}\lambda_- & -K\cos\,\gamma & 0 \\ N_+-1 & N_--1 & 1 & 1 \end{pmatrix}. \tag{21}$$

The condition $\sin(\theta_+-\theta_-)=\frac{2\omega}{K}$ for solutions says that $(\theta_+-\theta_-)$ may take two values: $\sin^{-1}\frac{2\omega}{K}$ and $\pi-\sin^{-1}\frac{2\omega}{K}$. In the first case $\cos\,\gamma>0$ and in the second case $\cos\,\gamma<0$. Therefore we must take into account these two possibilities when we look for stable solutions. It is clear that if $\cos\,\gamma>0$ then all eigenvalues of Λ are negative (positive) for $K>0$ ($K<0$) and therefore the solution is stable (unstable) for $K>0$ ($K<0$). If $\cos\,\gamma<0$ then for $K>0$ there is always a positive eigenvalue and the solution is unstable. For $K<0$ if we want all eigenvalues to be negative or zero then we should have $\cos\,\gamma=-1$. This is possible only if $\omega=0$ which is not the case in bimodal frequency distribution. Therefore for $\cos\,\gamma<0$ there is no stable solution.

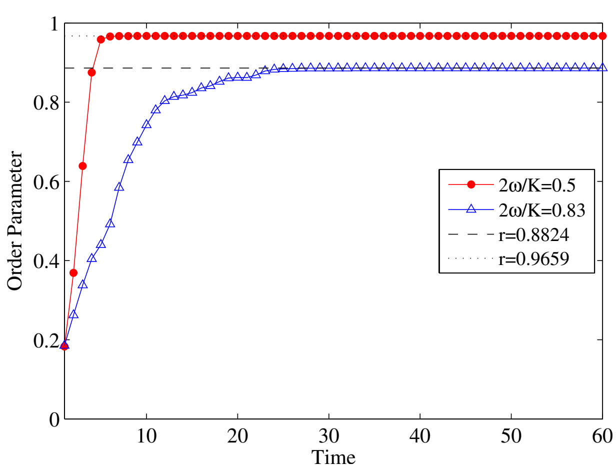

FIG. 2. (Color online) Time evolution of order parameter in an all-to-all network with 100 nodes and bimodal frequency distribution with $\omega=5$ and $\alpha=0.6$ for two values of $\sin\,\gamma$.

B. Bipartite network

In this section we consider only a *complete bipartite* network. In a complete bipartite network each node of one part is connected to all nodes of the other part. For this network the equations of motion take the form

$$\theta_i = \begin{cases} \omega + \frac{K}{N}\sum_{j=N_++1}^{N}\sin(\theta_j-\theta_i); & 1\le i\le N_+ \\ -\omega + \frac{K}{N}\sum_{j=1}^{N_+}\sin(\theta_j-\theta_i); & N_+<i\le N \end{cases}. \tag{22}$$

Then defining two partial order parameters r_+ and r_-,

$$r_+e^{i\psi_+} = \frac{1}{N_+}\sum_{j=1}^{N_+}e^{i\theta_j}, \quad r_-e^{i\psi_-} = \frac{1}{N_-}\sum_{j=N_++1}^{N}e^{i\theta_j}, \tag{23}$$

we arrive at

$$\Omega = \begin{cases} \omega + (1-\alpha)Kr_-\sin(\psi_--\theta_i) & 1\le i\le N_+ \\ -\omega+\alpha Kr_+\sin(\psi_+-\theta_i) & N_+<i\le N \end{cases}. \tag{24}$$

Then we get

$$\theta_i = \begin{cases} \theta_+ & 1\le i\le N_+ \\ \theta_- & N_+<i\le N \end{cases}, \tag{25}$$

where $\sin(\theta_+-\theta_-)=\frac{2\omega}{K}$. Using Lemma 1 of the Appendix and the adjacency matrix of a bipartite network, Eq. (12), it can be seen that for $K\cos\,\gamma>0$ ($K\cos\,\gamma<0$) the solutions are stable (unstable). The order parameter is clearly obtained from Eq. (19). In Fig. 3 the time evolution of the order parameter is shown for two values of $\sin\,\gamma$ which is in agreement with Eq. (19). It should be noted that a bipartite network has many other stable *solutions* (this is the subject of further investigations), but not all of them correspond to a *synchronized* state as introduced in this paper. Therefore in running the computer program one may arrive at different final (not necessarily synchronized) states depending on the choice of initial values of phases.

C. Semibipartite network

In this subsection we consider a complete semibipartite network. In a complete semibipartite network each peripheral node is connected to all central nodes. We assume that nodes $1,2,\ldots,N_c$ are central and nodes $N_c+1,\ldots,N$ are peripheral. For this network the equations of motion take the form

$$\theta_i = \begin{cases} \omega + \frac{K}{N}\sum_{j=1}^{N} \sin(\theta_j - \theta_i); & 1 \le i \le N_c \\ -\omega + \frac{K}{N}\sum_{j=1}^{N_c} \sin(\theta_j - \theta_i); & N_c < i \le N \end{cases}. \quad (26)$$

Then defining central and peripheral order parameters r_c and r_p,

$$r_c e^{i\psi_c} = \frac{1}{N_c}\sum_{j=1}^{N_c} e^{i\theta_j}, \quad r_p e^{i\psi_p} = \frac{1}{N_p}\sum_{j=N_c+1}^{N} e^{i\theta_j}, \quad (27)$$

we arrive at

$$\Omega = \begin{cases} \omega + Kr\sin(\psi\theta_i) & 1 \le i \le N_c \\ -\omega + \alpha K r_c \sin(\psi_c - \theta_i) & N_c < i \le N \end{cases}. \quad (28)$$

Then we get

$$\theta_i = \begin{cases} \theta_+ & 1 \le i \le N_c \\ \theta_- & N_c < i \le N \end{cases}, \quad (29)$$

where $\sin(\theta_+ - \theta_-) = \frac{2\omega}{K}$. The matrix Λ is

$$\Lambda = \frac{K}{N}\begin{pmatrix} J - \lambda_c I & \cos\gamma L \\ \cos\gamma L^t & -\lambda_p I \end{pmatrix}, \quad (30)$$

where $\lambda_c = (N_c + N_p \cos\gamma)$ and $\lambda_p = N_c \cos\gamma$. The spectrum of Λ is

$$\mathrm{Spec}(\Lambda) = \begin{pmatrix} -\frac{K}{N}\lambda_c & -\frac{K}{N}\lambda_p & -K\cos\gamma & 0 \\ N_c - 1 & N_p - 1 & 1 & 1 \end{pmatrix}. \quad (31)$$

Again for $\cos\gamma > 0$ the solution is always stable (unstable) for $K>0$ ($K<0$). For $\cos\gamma < 0$ if $K>0$ there is a positive eigenvalue and the solution is unstable. If $K<0$ then to have all eigenvalues nonpositive the condition $\cos\gamma < -\frac{N_c}{N_p}$ must be satisfied. This can be satisfied only if the number of peripheral nodes is greater than the number of central nodes. For stable solutions the final order parameter is obtained from Eq. (19). Figure 4 shows the time evolution of the order parameter for three values of coupling constants. The starting point of the order parameter is 1. This is because of the choice of initial values. The important point is that in stable cases the final values agree with Eq. (19).

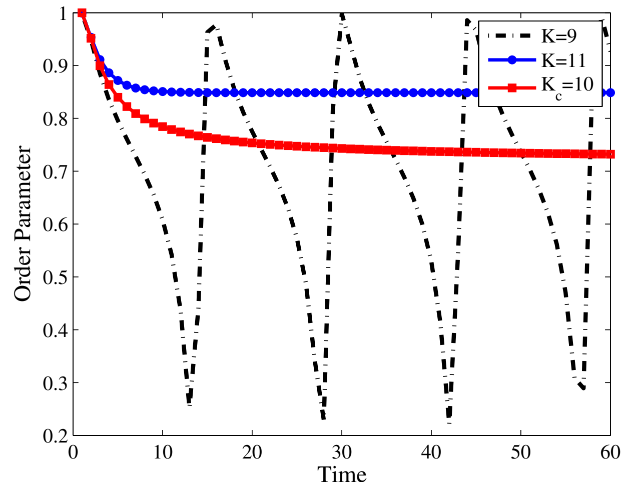

FIG. 4. (Color online) Time evolution of order parameter in a semibipartite network with 100 nodes (40 central nodes) and bimodal frequency distribution with $\omega=5$ and $\alpha=0.4$ for three values of coupling constant.

V. SUMMARY AND CONCLUSIONS

In this paper we studied the stability of synchronized states (which are defined as frequency locked states in this paper) in three kinds of networks of phase oscillators, namely complete (all-to-all) network, bipartite network, and semibipartite network. We considered two kinds of frequency distributions: unimodal and bimodal delta function distributions. We showed that according to the sign of coupling strength between oscillators which may be interpreted as attracting or repelling force, there may exist various stable synchronized states. In some of the stable solutions the order parameter is zero. In the case of bipartite networks a similarity between stable solutions for different signs of coupling strength and FM and AFM states in magnetic systems is addressed. Our results on stability are independent of the size of the network, namely the number of oscillators N. Among the networks investigated in this paper, complete bipartite and complete semibipartite networks may be considered as extreme cases of networks which can be distinguished by a parameter called *bipartivity parameter* and the path to a synchronized state may depend on this parameter. In Ref. [19] bipartivity of networks is investigated and a bipartivity parameter is introduced. In this paper we did not investigate the stability of the synchronized state in scale-free networks which are of considerable importance in network theory. This is the subject of further investigations. The results are physically reach and with many different features which will be published elsewhere [20].

APPENDIX

In this Appendix some mathematical relations used in the paper are summarized in two lemmas.

Lemma 1. Suppose that M is a square $N\times N$ symmetric matrix with non-negative elements. Then the eigenvalues of $D-M$ are non-negative where $D=\mathrm{diag}(d_1,d_2,\ldots,d_N)$ with $d_i=\sum_{j=1}^{N} M_{ij}$.

Proof. Construct a weighted network with N nodes. Give weight $\sqrt{M_{ij}}$ to link (ij) (note that in general a node may have a link with itself). Give an arbitrary direction to links. Construct the incidence matrix B as follows: B is a $N\times L$ matrix where $L=\frac{N(N-1)}{2}$ is the number of links. The element B_{iJ} is zero if node i is not any end point of the of link J. It is $\sqrt{M_J}$ if node i is the beginning point of link J and $-\sqrt{M_J}$ if it is the end point of link J. It is easily seen that $D-M = BB^t$. In this form it is clear that the eigenvalues of $D-M$ are non-negative.

Lemma 2. Consider a matrix Λ of the form

$$\Lambda = \begin{pmatrix} (aJ+bI)_M & kL_{M\times N} \\ kL^t_{N\times M} & (cJ+dI)_N \end{pmatrix}, \tag{A1}$$

where J is a square matrix whose all elements are 1, I is the identity matrix, and L is an $M\times N$ matrix whose all elements are 1. a, b, c, d, and k are real parameters. Then the eigenvalues of Λ are

$$\mathrm{Spec}(\Lambda) = \begin{pmatrix} b & d & \lambda_+ & \lambda_- \\ M-1 & N-1 & 1 & 1 \end{pmatrix}, \tag{A2}$$

where $\lambda_\pm = \frac{1}{2}(t_+ \pm \sqrt{t_-^2 + 4MNk^2})$ with $t_\pm = (Ma+b)\pm(Nc+d)$.

Proof. Eigenvalues are the roots of the characteristic equation $\det(\Lambda-\lambda I)=0$. With the help of properties of the determinant which is invariant under some matrix operation it is straightforward (although perhaps lengthy) to show directly that the eigenvalues of Eq. (A1) are given by Eq. (A2).

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